8.8 Indeterminate forms and L'Hôpital's Rule

SECTION TOPICS
- Indeterminate forms
- L'Hôpital's Rule

In Chapter 2, we described the forms 0/0, ∞/∞, and ∞ − ∞ as *indeterminate*, because they do not guarantee that a limit exists, nor do they indicate what the limit is, if one exists. When we encountered one of these indeterminate forms, we attempted to rewrite the expression by using various algebraic techniques, as illustrated by the examples in Table 8.1.

TABLE 8.1 INDETERMINATE FORMS

0/0	$\lim_{x\to-1}\frac{2x^2-2}{x+1}=\lim_{x\to-1}2(x-1)=-4$	Divide numerator and denominator by $(x+1)$.
∞/∞	$\lim_{x\to\infty}\frac{3x^2-1}{2x^2+1}=\lim_{x\to\infty}\frac{3-(1/x^2)}{2+(1/x^2)}=\frac{3}{2}$	Divide numerator and denominator by x^2.
∞ − ∞	$\lim_{x\to\infty}(x-\sqrt{x^2+x})=\lim_{x\to\infty}\frac{-x}{x+\sqrt{x^2+x}}$ $=\lim_{x\to\infty}\frac{-1}{1+\sqrt{1+(1/x)}}=-\frac{1}{2}$	Rationalize: Then divide numerator and denominator by x.

Occasionally, we can extend these algebraic techniques to find limits of transcendental functions. For instance, the limit

$$\lim_{x\to0}\frac{e^{2x}-1}{e^x-1}$$

produces the indeterminate form 0/0. Factoring and then dividing, we have

$$\lim_{x\to0}\frac{e^{2x}-1}{e^x-1}=\lim_{x\to0}\frac{(e^x+1)(e^x-1)}{e^x-1}$$
$$=\lim_{x\to0}(e^x+1)=2$$

However, not all indeterminate forms can be evaluated by algebraic manipulation. This is particularly true when *both* algebraic and transcendental functions are involved. For instance, the limit

$$\lim_{x\to0}\frac{e^{2x}-1}{x}$$

produces the indeterminate form 0/0. Dividing the numerator and denominator by x to obtain

$$\lim_{x\to0}\left[\frac{e^{2x}}{x}-\frac{1}{x}\right]$$

merely produces another indeterminate form, ∞ − ∞. Of course, we could use a calculator to estimate this limit, as shown in Table 8.2. From the table, the limit appears to be 2. (This limit will be verified in Example 1.)

Guillaume L'Hôpital

TABLE 8.2

x	-1.0	-0.1	-0.01	-0.001	0	0.001	0.01	0.1	1.0
$\dfrac{e^{2x}-1}{x}$	0.865	1.813	1.980	1.998	?	2.002	2.020	2.214	6.389

To find the limit illustrated in Table 8.2, we introduce a theorem called **L'Hôpital's Rule.** This theorem states that under certain conditions the limit of the quotient $f(x)/g(x)$ is determined by the limit of $f'(x)/g'(x)$. This theorem is named after the French mathematician Guillaume Francois Antoine De L'Hôpital (1661–1704), who published the first calculus text in 1696. To prove this theorem, we use a more general result called the **Extended Mean-Value Theorem,** which states the following. If f and g are differentiable on an open interval (a, b) and continuous on $[a, b]$ such that $g(x) \neq 0$ for any x in (a, b), then there exists a point c in (a, b) such that

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

We prove the Extended Mean-Value Theorem and L'Hôpital's Rule in Appendix A.

Remark In the following theorem, we use the symbol "lim" to represent any of the following types of limits:

$$\lim_{x\to a} \qquad \lim_{x\to a^+} \qquad \lim_{x\to a^-} \qquad \lim_{x\to\infty} \qquad \lim_{x\to-\infty}$$

THEOREM 8.7 L'HÔPITAL'S RULE

If $\lim f(x)/g(x)$ results in the indeterminate form 0/0 or ∞/∞,* then

$$\lim \frac{f(x)}{g(x)} = \lim \frac{f'(x)}{g'(x)}$$

provided the latter limit exists (or is infinite).

EXAMPLE 1 Indeterminate form 0/0

Evaluate

$$\lim_{x\to 0} \frac{e^{2x}-1}{x}$$

Solution: Since direct substitution results in the indeterminate form 0/0, we apply L'Hôpital's Rule to obtain

$$\lim_{x\to 0} \frac{e^{2x}-1}{x} = \lim_{x\to 0} \frac{\dfrac{d}{dx}[e^{2x}-1]}{\dfrac{d}{dx}[x]} = \lim_{x\to 0} \frac{2e^{2x}}{1} = 2$$

*The indeterminate form ∞/∞ actually comes in four forms: ∞/∞, $(-\infty)/\infty$, $\infty/(-\infty)$, and $(-\infty)/(-\infty)$. L'Hôpital's Rule can be applied to each of these forms.

Remark In writing the string of equations in Example 1, we actually do not know that the first limit is equal to the second until we have shown that the second limit exists. In other words, if the second limit had not existed, it would not have been permissible to apply L'Hôpital's Rule.

Occasionally it is necessary to apply L'Hôpital's Rule more than once to remove an indeterminate form. This is illustrated in the next example.

EXAMPLE 2 Indeterminate form ∞/∞

Evaluate

$$\lim_{x\to-\infty} \frac{x^2}{e^{-x}}$$

Solution: Since direct substitution results in the indeterminate form ∞/∞, we apply L'Hôpital's Rule to obtain

$$\lim_{x\to-\infty} \frac{x^2}{e^{-x}} = \lim_{x\to-\infty} \frac{\dfrac{d}{dx}[x^2]}{\dfrac{d}{dx}[e^{-x}]} = \lim_{x\to-\infty} \frac{2x}{-e^{-x}}$$

Since this limit yields the indeterminate form $(-\infty)/(-\infty)$, we apply L'Hôpital's Rule again to obtain

$$\lim_{x\to-\infty} \frac{2x}{-e^{-x}} = \lim_{x\to-\infty} \frac{\dfrac{d}{dx}[2x]}{\dfrac{d}{dx}[-e^{-x}]} = \lim_{x\to-\infty} \frac{2}{e^{-x}} = 0$$

In addition to the forms $0/0$, ∞/∞, and $\infty - \infty$, there are other indeterminate forms, such as $0 \cdot \infty$, 1^∞, ∞^0, and 0^0. For example, consider the following four limits that lead to the indeterminate form $0 \cdot \infty$:

$$\underbrace{\lim_{x\to0} (x)\left(\frac{1}{x}\right)}_{\text{limit is 1}}, \quad \underbrace{\lim_{x\to0} (x)\left(\frac{2}{x}\right)}_{\text{limit is 2}}, \quad \underbrace{\lim_{x\to\infty} (x)\left(\frac{1}{e^x}\right)}_{\text{limit is 0}}, \quad \underbrace{\lim_{x\to\infty} (e^x)\left(\frac{1}{x}\right)}_{\text{limit is }\infty}$$

Since each limit is different, it is clear that the form $0 \cdot \infty$ is indeterminate in the sense that it does not determine the value (or even the existence) of the limit. The following examples indicate methods to evaluate these forms. Basically, we attempt to convert each of these forms to those for which L'Hôpital's Rule is applicable.

EXAMPLE 3 Indeterminate form $0 \cdot \infty$

Evaluate

$$\lim_{x\to\infty} e^{-x}\sqrt{x}$$

Solution: Since direct substitution produces the indeterminate form $0 \cdot \infty$, we rewrite the limit to fit the form $0/0$ or ∞/∞. In this case, we choose the second form and write

$$\lim_{x\to\infty} e^{-x}\sqrt{x} = \lim_{x\to\infty} \frac{\sqrt{x}}{e^x}$$

Now, by L'Hôpital's Rule, we have

$$\lim_{x\to\infty} \frac{\sqrt{x}}{e^x} = \lim_{x\to\infty} \frac{1/(2\sqrt{x})}{e^x} = \lim_{x\to\infty} \frac{1}{2\sqrt{x}\, e^x} = 0$$

Remark If rewriting a limit in one of the forms $0/0$ or ∞/∞ does not seem to work, try the other form. For instance, in Example 3 we could have written the limit as

$$\lim_{x\to\infty} e^{-x}\sqrt{x} = \lim_{x\to\infty} \frac{e^{-x}}{x^{-1/2}}$$

which yields the indeterminate form $0/0$. However, in applying L'Hôpital's Rule to this limit, we obtain

$$\lim_{x\to\infty} \frac{e^{-x}}{x^{-1/2}} = \lim_{x\to\infty} \frac{-e^{-x}}{-1/(2x^{3/2})}$$

which also yields the indeterminate form $0/0$. Moreover, since the quotient seems to be getting more complicated, we abandon this approach and try the ∞/∞ form, as shown in Example 3.

The indeterminate forms 1^∞, ∞^0, and 0^0 arise from limits of functions that have a variable base and a variable exponent. When we encountered this type of function in Section 6.2, we used logarithmic differentiation to find the derivative. We use a similar procedure when taking limits, as indicated in the next example.

EXAMPLE 4 Indeterminate form 1^∞

Evaluate

$$\lim_{x\to\infty} \left(1 + \frac{2}{x}\right)^x$$

Solution: Since direct substitution yields the indeterminate form 1^∞, we proceed in the following manner. We assume the limit exists, and we represent it by

$$y = \lim_{x\to\infty} \left(1 + \frac{2}{x}\right)^x$$

Now, taking the natural logarithm of both sides, we have

$$\ln y = \ln \left[\lim_{x\to\infty} \left(1 + \frac{2}{x}\right)^x\right]$$

and using the fact that the natural logarithmic function is continuous, we write

$$\ln y = \lim_{x\to\infty}\left[x \ln\left(1 + \frac{2}{x}\right)\right] \qquad \text{Indeterminate form: } \infty \cdot 0$$

$$= \lim_{x\to\infty}\left[\frac{\ln[1 + (2/x)]}{1/x}\right] \qquad \text{Indeterminate form: } 0/0$$

$$= \lim_{x\to\infty}\left[\frac{(-2/x^2)(1/[1 + (2/x)])}{-1/x^2}\right] \qquad \text{L'Hôpital's Rule}$$

$$= \lim_{x\to\infty}\frac{2}{1 + (2/x)} = 2$$

Finally, since $\ln y = 2$, we know that $y = e^2$ and we conclude that

$$\lim_{x\to\infty}\left(1 + \frac{2}{x}\right)^x = e^2$$

EXAMPLE 5 Indeterminate form 0^0

Evaluate

$$\lim_{x\to 0^+}(\sin x)^x$$

Solution: Since direct substitution produces the indeterminate form 0^0, we proceed as follows.

$$y = \lim_{x\to 0^+}(\sin x)^x \qquad \text{Indeterminate form } 0^0$$

$$\ln y = \ln\left[\lim_{x\to 0^+}(\sin x)^x\right] \qquad \text{Take log of both sides}$$

$$= \lim_{x\to 0^+}[\ln(\sin x)^x] \qquad \text{Continuity}$$

$$= \lim_{x\to 0^+}[x\ln(\sin x)] \qquad \text{Indeterminate form } 0 \cdot (-\infty)$$

$$= \lim_{x\to 0^+}\frac{\ln(\sin x)}{1/x} \qquad \text{Indeterminate form } -\infty/\infty$$

$$= \lim_{x\to 0^+}\frac{\cot x}{-1/x^2} \qquad \text{L'Hôpital's Rule}$$

$$= \lim_{x\to 0^+}\frac{-x^2}{\tan x} \qquad \text{Indeterminate form } 0/0$$

$$= \lim_{x\to 0^+}\frac{-2x}{\sec^2 x} = 0 \qquad \text{L'Hôpital's Rule}$$

Now, since $\ln y = 0$, we conclude that $y = e^0 = 1$, and it follows that

$$\lim_{x\to 0^+}(\sin x)^x = 1$$

Remark When evaluating complicated limits like the one in Example 5, it is helpful to check the reasonableness of the solution with a calculator. For instance, the calculations shown in Table 8.3 are consistent with our conclusion that $(\sin x)^x$ approaches 1 as x approaches 0 from the right.

TABLE 8.3

x	1.0	0.1	0.01	0.001	0.0001	0.00001
$(\sin x)^x$	0.8415	0.7942	0.9550	0.9931	0.9991	0.9999

EXAMPLE 6 Indeterminate form $\infty - \infty$

Evaluate

$$\lim_{x\to1^+}\left(\frac{1}{\ln x}-\frac{1}{x-1}\right)$$

Solution: Since direct substitution yields the indeterminate form $\infty - \infty$, we try to rewrite the expression to produce a form to which we can apply L'Hôpital's Rule. In this case, we combine the two fractions to obtain

$$\lim_{x\to1^+}\left(\frac{1}{\ln x}-\frac{1}{x-1}\right)=\lim_{x\to1^+}\left[\frac{x-1-\ln x}{(x-1)\ln x}\right]$$

Now, since direct substitution produces the indeterminate form 0/0, we can apply L'Hôpital's Rule to obtain

$$\begin{aligned}\lim_{x\to1^+}\left(\frac{1}{\ln x}-\frac{1}{x-1}\right)&=\lim_{x\to1^+}\left[\frac{1-(1/x)}{(x-1)(1/x)+\ln x}\right]\\&=\lim_{x\to1^+}\left[\frac{x-1}{x-1+x\ln x}\right]\end{aligned}$$

This limit also yields the indeterminate form 0/0, so we apply L'Hôpital's Rule again to obtain

$$\lim_{x\to1^+}\left(\frac{1}{\ln x}-\frac{1}{x-1}\right)=\lim_{x\to1^+}\left[\frac{1}{1+x(1/x)+\ln x}\right]=\frac{1}{2}$$

Remark We have identified the forms $0/0$, ∞/∞, $\infty-\infty$, $0\cdot\infty$, 0^0, 1^∞, and ∞^0 as *indeterminate*. There are similar forms that you should recognize as *determinate*, such as

$$\infty+\infty\to\infty \qquad -\infty-\infty\to-\infty \qquad 0^\infty\to0 \qquad 0^{-\infty}\to\infty$$

In each of the examples so far in this section, we have used L'Hôpital's Rule to find a limit that exists. L'Hôpital's Rule can also be used to conclude that a limit is infinite and this is demonstrated in the last example.

EXAMPLE 7 An infinite limit

Evaluate

$$\lim_{x\to\infty}\frac{e^x}{x}$$

Solution: Direct substitution produces the indeterminate form ∞/∞, so we apply L'Hôpital's Rule to obtain

$$\lim_{x\to\infty} \frac{e^x}{x} = \lim_{x\to\infty} \frac{e^x}{1} = \lim_{x\to\infty} e^x = \infty$$

Now, since $e^x \to \infty$ as $x \to \infty$, we conclude that the limit of e^x/x as $x \to \infty$ is also infinite.

As a final comment, we remind you that L'Hôpital's Rule can only be applied to quotients leading to the indeterminate forms 0/0 or ∞/∞. For instance, the following application of L'Hôpital's Rule is *incorrect*.

$$\lim_{x\to 0} \frac{e^x}{x} = \lim_{x\to 0} \frac{e^x}{1} = 1 \qquad \text{Incorrect use of L'Hôpital's Rule}$$

The reason this application is incorrect is that, even though the limit of the denominator is 0, the limit of the numerator is 1—which means that the hypotheses of L'Hôpital's Rule have not been satisfied.

Section Exercises 8.8

1-45 odd

In Exercises 1–34, evaluate each limit, using L'Hôpital's Rule where necessary.

1. $\displaystyle\lim_{x\to 2} \frac{x^2 - x - 2}{x - 2}$

2. $\displaystyle\lim_{x\to -1} \frac{x^2 - x - 2}{x + 1}$

3. $\displaystyle\lim_{x\to 0} \frac{\sqrt{4 - x^2} - 2}{x}$

4. $\displaystyle\lim_{x\to 2^-} \frac{\sqrt{4 - x^2}}{x - 2}$

5. $\displaystyle\lim_{x\to 0} \frac{e^x - (1 - x)}{x}$

6. $\displaystyle\lim_{x\to 0^+} \frac{e^x - (1 + x)}{x^3}$

7. $\displaystyle\lim_{x\to 0^+} \frac{e^x - (1 + x)}{x^n}$, $n = 1, 2, 3, \ldots$

8. $\displaystyle\lim_{x\to 1} \frac{\ln x}{x^2 - 1}$

9. $\displaystyle\lim_{x\to\infty} \frac{\ln x}{x}$

10. $\displaystyle\lim_{x\to\infty} \frac{e^x}{x}$

11. $\displaystyle\lim_{x\to\infty} \frac{3x^2 - 2x + 1}{2x^2 + 3}$

12. $\displaystyle\lim_{x\to\infty} \frac{x - 1}{x^2 + 2x + 3}$

13. $\displaystyle\lim_{x\to\infty} \frac{x^2 + 2x + 3}{x - 1}$

14. $\displaystyle\lim_{x\to\infty} \frac{x^2}{e^x}$

15. $\displaystyle\lim_{x\to 0^+} x^2 \ln x$

16. $\displaystyle\lim_{x\to 0} \left(\frac{1}{x} - \frac{1}{x^2}\right)$

17. $\displaystyle\lim_{x\to 2} \left(\frac{8}{x^2 - 4} - \frac{x}{x - 2}\right)$

18. $\displaystyle\lim_{x\to 2} \left(\frac{1}{x^2 - 4} - \frac{\sqrt{x - 1}}{x^2 - 4}\right)$

19. $\displaystyle\lim_{x\to\infty} \frac{x}{\sqrt{x^2 + 1}}$

20. $\displaystyle\lim_{x\to 1^+} \left(\frac{3}{\ln x} - \frac{2}{x - 1}\right)$

21. $\displaystyle\lim_{x\to 0^+} x^{1/x}$

22. $\displaystyle\lim_{x\to 0^+} (e^x + x)^{1/x}$

23. $\displaystyle\lim_{x\to\infty} x^{1/x}$

24. $\displaystyle\lim_{x\to\infty} \left(1 + \frac{1}{x}\right)^x$

25. $\displaystyle\lim_{x\to\infty} (1 + x)^{1/x}$

26. $\displaystyle\lim_{x\to\pi} \frac{\sin x}{x - \pi}$

27. $\displaystyle\lim_{x\to 0} \frac{\sin 2x}{\sin 3x}$

28. $\displaystyle\lim_{x\to 0} \frac{\sin ax}{\sin bx}$

29. $\displaystyle\lim_{x\to 0} x \csc x$

30. $\displaystyle\lim_{x\to 0} x^2 \cot x$

31. $\displaystyle\lim_{x\to\infty} \left(x \sin \frac{1}{x}\right)$

32. $\displaystyle\lim_{x\to\infty} x \tan \frac{1}{x}$

33. $\displaystyle\lim_{x\to 0} \frac{\arcsin x}{x}$

34. $\displaystyle\lim_{x\to 1} \frac{\arctan x - (\pi/4)}{x - 1}$

In Exercises 35–40, use L'Hôpital's Rule to determine the comparative rates of increase of the functions

$$f(x) = x^m$$
$$g(x) = e^{nx}$$
$$h(x) = (\ln x)^n$$

where $n > 0$, $m > 0$, and $x \to \infty$. The limits obtained in these exercises suggest that $(\ln x)^n$ tends toward infinity more slowly than x^m, which in turn tends toward infinity more slowly than e^{nx}.

35. $\displaystyle\lim_{x\to\infty} \frac{x^2}{e^{5x}}$

36. $\displaystyle\lim_{x\to\infty} \frac{x^3}{e^{2x}}$

37. $\displaystyle\lim_{x\to\infty} \frac{(\ln x)^3}{x}$

38. $\displaystyle\lim_{x\to\infty} \frac{(\ln x)^2}{x^3}$